LIMIT OF THREE-POINT GREEN FUNCTIONS : THE DEGENERATE CASE

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Abstract. We investigate the limits of the ideals of holomorphic functions vanishing on three points in $\mathbb{C}^2$ when all three points tend to the origin, and what happens to the associated pluricomplex Green functions. This is a continuation of the work of Magnusson, Rashkovskii, Sigurdsson and Thomas, where those questions were settled in a generic case.

1. Introduction

Let Ω be a hyperconvex bounded domain in $\mathbb{C}^n$ containing the origin 0 and let $\mathcal{O}(\Omega)$ denote the space of holomorphic functions, respectively $PSH_-(\Omega)$ the space of nonpositive plurisubharmonic functions on Ω. For every subset S of Ω we let $\mathcal{I}(S)$ denote the ideal of all holomorphic functions vanishing on S. We consider ideals $\mathcal{I}$ such that their zero locus $V(\mathcal{I}) := \{z \in \Omega : f(z) = 0, \forall f \in \mathcal{I}\}$ is a finite set. Since the domain is pseudoconvex, there are finitely many global generators $\psi_j \in \mathcal{O}(\Omega)$ such that for any $f \in \mathcal{I}$, there exists $h_j \in \mathcal{O}(\Omega)$ such that $f = \sum_j h_j \psi_j$, see e.g. [2, Theorem 7.2.9, p. 190].

Definition 1.1. [6] *Let $\mathcal{I}$ be an ideal of Ω, and ψ_j its generators. Then*

$$G_{\mathcal{I}}^{\Omega}(z) := \sup \left\{ u(z) : u \in PSH_-(\Omega), u(z) \leq \max_j \log |\psi_j| + O(1) \right\}.$$

Note that the condition is meaningful only near $a \in V(\mathcal{I})$. In the special case when S is a finite set in Ω and $\mathcal{I} = \mathcal{I}(S)$, we write $G_{\mathcal{I}(S)} = G_S$. This case reduces to Pluricomplex Green functions with logarithmic singularities, already studied by many authors, e.g. Demailly [1], [7], Lelong [3], and Rashkovskii and Sigurdsson [6].

Following the lead of [4], we want to study the limit of G_{S_ε} when S_ε is a set of points tending to the origin, and relate that to the limit of the ideals $\mathcal{I}(S_\varepsilon)$ (in a sense to be specified below, see [4] for more details).

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Definition 1.2. *A (point based) ideal is a* complete intersection ideal *if and only if it admits a set of n generators, where n is the dimension of the ambient space.*

The main result of [4], Theorem 1.11, states:

Theorem 1.3. *Let $\mathcal{I}_\varepsilon = \mathcal{I}(S_\varepsilon)$, where S_ε is a set of N points all tending to 0 and assume that $\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon = \mathcal{I}$. Then $(G_{\mathcal{I}_\varepsilon})$ converges to $G_{\mathcal{I}}$ locally uniformly on $\Omega\setminus\{0\}$ if and only if $\mathcal{I}$ is a complete intersection ideal.*

Furthermore, [4, Theorem 1.12, (i)] works out the limits of Green functions when $N = 3$ and the dimension $n = 2$.

Let $S_\varepsilon := \{a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon\}$. For each pair of distinct indices, i, j, let $[a_i^\varepsilon - a_j^\varepsilon] = v_k^\varepsilon \in \mathbb{CP}^1$ where $\{i,j,k\} = \{1,2,3\}$. The cases which are studied in [4] are those where there exist $i \neq j$ such that $\lim_{\varepsilon\to 0} v_i^\varepsilon$ and $\lim_{\varepsilon\to 0} v_j^\varepsilon$ exist and are distinct. In those cases $\lim_{\varepsilon\to 0}\mathcal{I}(S_\varepsilon) = \mathfrak{M}_0^2$ (the square of the maximal ideal at zero, i.e. the set of functions vanishing at zero together with all their first derivatives), which is not a complete intersection ideal. This sufficient condition is not necessary. We will give a characterization of the situations where the limit equals the square of the maximal ideal at zero.

The main goal of this note is to investigate the asymptotic behavior of ideals and Green functions in the remaining (and most singular) case, when there exists $v \in \mathbb{C}^2$, with $\|v\| = 1$, such that

$$\lim_{\varepsilon\to 0} v_i^\varepsilon = [v] \text{ for } 1 \leq i \leq 3. \tag{1.1}$$

We use the notation $z\cdot\bar w := z_1\bar w_1 + z_2\bar w_2$ for $z, w \in \mathbb{C}^2$, and $\|z\|^2 := z\cdot\bar z$.

Numbering the points.

The notions we study do not depend on the order of the points in S_ε, nor does (1.1). We choose an appropriate numbering. Set $d_i^\varepsilon := ||a_j^\varepsilon - a_k^\varepsilon||$, the Euclidean distances between two of the three points, for $\{i,j,k\} = \{1,2,3\}$, $j,k \neq i$. For each ε, number the points so that $d_3^\varepsilon \geqslant d_1^\varepsilon \geqslant d_2^\varepsilon$.

We perform a translation so that $a_1^\varepsilon = (0,0)$. Since the distance from a_1^ε to the origin tends to 0 by hypothesis, this does not change any of the limits we are studying, and we shall make this assumption henceforth.

Let θ be the acute angle between the lines directed by a_2^ε and a_3^ε, i.e. $\theta := \cos^{-1}\left(\frac{|a_2^\varepsilon\cdot\bar a_3^\varepsilon|}{\|a_2^\varepsilon\|\|a_3^\varepsilon\|}\right)$.

Theorem 1.4. $\lim_{\varepsilon\to 0}\mathcal{I}(S_\varepsilon)=\mathfrak{M}_0^2$ *if and only if* $\lim_{\varepsilon\to 0}\frac{\|a_2^\varepsilon\|}{|\theta|}=0$, *or equivalently*

$$\lim_{\varepsilon\to 0}\frac{\|a_2^\varepsilon\|}{|\det(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|},\frac{a_3^\varepsilon}{\|a_3^\varepsilon\|})|}=0, \tag{1.2}$$

where the determinant is taken with respect to an orthonormal basis.

The condition can be rephrased in a coordinate-free way using some elementary plane geometry: let d be the diameter of the set S_ε, let $\theta_1\le\theta_2\le\theta_3$ be the three angles (in $[0,\pi]$) determined by the triangle $a_1a_2a_3$; then we require that d/θ_2 tend to 0.

Choice of coordinates.

Now suppose that all three points converge to the origin along a common direction, that is, that there exists v such that (1.1) holds. We reparametrize our family so that $|\varepsilon|=\|a_1^\varepsilon-a_2^\varepsilon\|$. We choose coordinates depending on ε so that $a_2^\varepsilon=(\varepsilon,0), a_3^\varepsilon=(\rho_1(\varepsilon),\rho_2(\varepsilon))$, where $\lim_{\varepsilon\to 0}\rho_j(\varepsilon)=0$ for $j=1,2$, and $\lim_{\varepsilon\to 0}\dfrac{\rho_2(\varepsilon)}{\rho_1(\varepsilon)}=0$ (see details before (2.1)).

Furthermore, $a_3^\varepsilon\in B(0;|\varepsilon|)\cap B(a_2^\varepsilon;|\varepsilon|)$ so we have

$$|\rho_1(\varepsilon)|\leqslant\frac{1}{2}|\varepsilon|, |\rho_2(\varepsilon)|\leqslant\frac{\sqrt{3}}{2}|\varepsilon|.$$

Finally, we can write $a_3^\varepsilon=(\rho(\varepsilon),\delta(\varepsilon)\rho(\varepsilon))$, with $0\neq|\rho(\varepsilon)|\leqslant\dfrac{1}{2}|\varepsilon|$, $\delta(\varepsilon)\to 0$.

Theorem 1.5. *Under the above hypotheses:*

(1) *If* $\lim\limits_{\varepsilon\longrightarrow 0}\dfrac{\delta(\varepsilon)}{\rho(\varepsilon)-\varepsilon}=m\neq\infty$, *then*

$$\lim_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon=\mathcal{I}=\left\langle z_2-mz_1^2,z_1^3\right\rangle,$$

so by Theorem 1.3,

$$\lim_{\varepsilon\longrightarrow 0}G_\varepsilon(z)=G_{\mathcal{I}}(z)=\max\big(\log|z_2-mz_1^2|,3\log|z_1|\big).$$

(2) *If* $\lim\limits_{\varepsilon\longrightarrow 0}\dfrac{\delta(\varepsilon)}{\rho(\varepsilon)-\varepsilon}=\infty$ *(equivalently* $\lim\limits_{\varepsilon\longrightarrow 0}\dfrac{\delta(\varepsilon)}{\varepsilon}=\infty$*), then*

$$\lim_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon=\mathfrak{M}_0^2,$$

so by Theorem 1.3, the Green function cannot converge to $G_{\mathfrak{M}_0^2}(z)=2\log\|z\|+O(1)$.

We still need to understand to which limit the Green function may converge in case (2), at least in the model case where $\Omega = \mathbb{D}^2$. Unfortunately, we could only get some partial estimates.

Proposition 1.6. *(1) For any* $z = (z_1, z_2) \in \mathbb{D}^2 \setminus \{(0,0)\}$,

$$\liminf_{\varepsilon \longrightarrow 0} G_\varepsilon(z_1, z_2) \geqslant \max\left(2 \log |z_1|, \frac{3}{2} \log |z_2|\right).$$

(2) For any $z = (z_1, z_2) \in \mathbb{D}^2 \setminus \{(0,0)\}$ *such that* $|z_2| \leqslant |z_1|^2$,

$$\lim_{\varepsilon \longrightarrow 0} G_\varepsilon(z_1, z_2) = 2 \log |z_1|.$$

In general, it is more difficult to get upper than lower bounds on the limits of Green functions. We only could get a general upper bound under rather special hypotheses on the configuration, roughly speaking that the angle with vertex at the origin formed by the three points should tend to 0 very slowly.

Proposition 1.7. *If* $\dfrac{\log |\delta|}{\log |\varepsilon|} \longrightarrow 0$ *as* $\varepsilon \to 0$*, then for any* $z = (z_1, z_2) \in \mathbb{D}^2 \setminus \{(0,0)\}$

$$\limsup_{\varepsilon \longrightarrow 0} G_\varepsilon(z) \leqslant \frac{3}{2} \log \max(|z_1|, |z_2|).$$

Corollary 1.8. *Under the hypotheses of the previous Proposition, if* $|z_2| \geq |z_1|$*, then* $\lim_{\varepsilon \to 0} G_\varepsilon(z) = \dfrac{3}{2} \log |z_2|$.

2. Proof of Theorems 1.4 and 1.5

2.1. **Preliminary facts.** First we need a notion of convergence of ideals, inspired by Hausdorff convergence. This is taken from [4]. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $E \subset \mathbb{C}$ such that $\bar{E} \ni 0$ the set of parameters along which we take limits. Convergence of holomorphic functions is always understood uniformly on compacta.

Definition 2.1. *If* $(\mathcal{I}_\varepsilon)_{\varepsilon \in E}$ *are ideals in* $\mathcal{O}(\Omega)$*, we define*

$$\liminf_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon := \{ f \in \mathcal{O}(\Omega) : \text{ for all } \varepsilon \in E, \exists f_\varepsilon \in \mathcal{I}_\varepsilon, f_\varepsilon \longrightarrow f \text{ when } \varepsilon \longrightarrow 0 \}.$$

Likewise $\limsup_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon$ *is the vector space generated by*

$$\{ f \in \mathcal{O}(\Omega) : \exists (\varepsilon_j)_{j \in \mathbb{Z}_+} \subset E, \varepsilon_j \longrightarrow 0 \text{ when } j \longrightarrow \infty; \exists f_j \in \mathcal{I}_{\varepsilon_j}, f_j \longrightarrow f \text{ when } j \longrightarrow \infty \}.$$

We say that $(\mathcal{I}_\varepsilon)_{\varepsilon\in E}$ *converges to* $\mathcal{I}$ *if and only if* $\liminf\limits_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon = \limsup\limits_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon = \mathcal{I}$*, and write* $\lim\limits_{\varepsilon\longrightarrow 0,\varepsilon\in E}\mathcal{I}_\varepsilon = \mathcal{I}$.

Of course $\liminf\limits_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon \subset \limsup\limits_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon$, and they are both ideals.

Denote the Taylor expansion and Taylor polynomial of a holomorphic function f by

$$f(z) = f(z_1,z_2) = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k; \quad P_m(f)(z) := \sum_{\substack{j,k\\ j+k\leqslant m}} a_{jk} z_1^j z_2^k.$$

It follows from the Cauchy formula on the distinguished boundary of $\mathbb{D}^2$ that

Lemma 2.2. *Let* $m\in\mathbb{N}^*$*,* U *a bidisk centered at* $(0,0)$*, relatively compact in* $\mathbb{D}^2$*. There exists* $C = C(m,U)$ *such that for any* $f\in\mathcal{O}(\mathbb{D}^2)$ *with* $\sup\limits_{\mathbb{D}^2}||f||\leqslant 1$*, there exist holomorphic functions* $r_{j,k}\in\mathcal{O}(\mathbb{D}^2)$ *satisfying : for* $j+k=m+1$*,* $\sup\limits_U|r_{j,k}|\leqslant C, 0\leqslant j\leqslant m+1$ *and for* $z=(z_1,z_2)\in U$*, then*

$$f(z) = P_m(f)(z) + R_{m+1}(z) = P_m(f)(z) + \sum_{j=0}^{m+1} r_{j,m+1-j}(z) z_1^j z_2^{m+1-j}.$$

2.2. **Proof of the sufficiency in Theorem 1.4.** Suppose that $f\in\limsup_\varepsilon\mathcal{I}_\varepsilon$. This means that there exists some subset $E\subset\mathbb{C}$ such that $0\in\overline{E}$ and a family of holomorphic functions $\{f^\varepsilon,\varepsilon\in E\}$, with $f^\varepsilon\in\mathcal{I}_\varepsilon$, $\varepsilon\in E$, converging to f uniformly on a fixed neighborhood U of the origin. Observe that all the Taylor coefficients will have to converge.

Since $f^\varepsilon(a_1^\varepsilon)=0$, $a_{0,0}^\varepsilon=0$ for any ε. Applying Lemma 2.2 for $m=1$, if $U\Subset U'\Subset\Omega$,

$$f^\varepsilon(z_1,z_2) = a_{1,0}^\varepsilon z_1 + a_{0,1}^\varepsilon z_2 + R_2(z_1,z_2)$$

with $|R_2(z_1,z_2)|\leq C\|z\|^2$, where C only depends on U, U' and $\sup_{U'}|f^\varepsilon|$.

Applying this to $z=a_i^\varepsilon$, dividing by $\|a_i^\varepsilon\|$ and writing $\nabla f^\varepsilon(0) := (a_{1,0}^\varepsilon, a_{0,1}^\varepsilon)$, we find

$$\frac{a_i^\varepsilon}{\|a_i^\varepsilon\|}\cdot\nabla f^\varepsilon(0) = O(\|a_i^\varepsilon\|),\quad i=2,3.$$

Write M for the 2×2 matrix with rows given by the coordinates of $\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}$ and $\frac{a_3^\varepsilon}{\|a_3^\varepsilon\|}$. Then $\|M\| = O(1)$ and $\|M^{-1}\| = O\left(|\det(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|},\frac{a_3^\varepsilon}{\|a_3^\varepsilon\|})|^{-1}\right)$.

Since by our choice of numbering, $\|a_3^\varepsilon\| \le \|a_2^\varepsilon\|$, we have

$$\nabla f^\varepsilon(0) = O(\|M^{-1}\|\|a_2^\varepsilon\|),$$

so that if condition (1.2) is met, then $\lim_{E\ni\varepsilon\to 0} \nabla f^\varepsilon(0) = 0$, thus $f \in \mathfrak{M}_0^2$. We have proved that condition (1.2) implies that $\limsup_\varepsilon \mathcal{I}_\varepsilon \subset \mathfrak{M}_0^2$.

To prove the inclusion $\mathfrak{M}_0^2 \subset \liminf_\varepsilon \mathcal{I}_\varepsilon$, it will be easier to take suitable coordinates.

For each pair of distinct indices, i, j, let $a_i^\varepsilon - a_j^\varepsilon = u_k^\varepsilon \in \mathbb{C}^2$ where $\{i,j,k\} = \{1,2,3\}$. For $|\varepsilon|$ small enough, $\theta \neq 0$ so $\{u_3^\varepsilon, u_2^\varepsilon\}$ are linearly independent. Using the Gram-Schmidt orthogonalization process, we get $\mathfrak{B}_\varepsilon := \{e_1^\varepsilon, e_1^\varepsilon\}$ an orthonormal basis of $\mathbb{C}^2$, where $e_j^\varepsilon = \dfrac{v_j^\varepsilon}{||v_j^\varepsilon||}$, for $j = 1,2$, $v_1^\varepsilon := u_3^\varepsilon$ and $v_2^\varepsilon := u_2^\varepsilon - \dfrac{u_2^\varepsilon.\overline{u_3^\varepsilon}}{||u_3^\varepsilon||^2}.u_3^\varepsilon$. If $z = (z_1, z_2) \in \Omega$ its new coordinates $(z_1^\varepsilon, z_2^\varepsilon)$ with respect to $\mathfrak{B}_\varepsilon$ are given by

$$z_1^\varepsilon = z \cdot \bar{e}_1^\varepsilon, z_2^\varepsilon = z \cdot \bar{e}_2^\varepsilon. \tag{2.1}$$

Denote the coordinates of the points in this new basis as before. Theorem 1.5 (condition (1.2) implies that the angle between v_2^ε and v_3^ε tends to 0, but no convergence of the basis is needed). Then $|\delta(\varepsilon)| = \tan\theta$, so our hypothesis implies that $\lim \varepsilon/\delta = \lim(\rho(\varepsilon) - \varepsilon)/\delta = 0$.

The following polynomials are in $\mathcal{I}_\varepsilon$:

$$\begin{aligned}
Q_1^\varepsilon(z) &= (z_1^\varepsilon)^2 - \varepsilon z_1^\varepsilon - \frac{\rho - \varepsilon}{\delta} z_2^\varepsilon;\\
Q_2^\varepsilon(z) &= z_2^\varepsilon\big(z_1^\varepsilon - \rho\big);\\
Q_3^\varepsilon(z) &= z_2^\varepsilon\big(z_2^\varepsilon - \delta\rho\big).
\end{aligned}$$

Let $\alpha_{ij} := e_j^\varepsilon \cdot \bar{e}_i$, for $1 \leqslant i,j \leqslant 2$, so that

$$\begin{aligned}
z_1 &= \alpha_{11} z_1^\varepsilon + \alpha_{12} z_2^\varepsilon,\\
z_2 &= \alpha_{21} z_1^\varepsilon + \alpha_{22} z_2^\varepsilon.
\end{aligned}$$

If we let

$$\begin{aligned}
f_1^\varepsilon(z) &= \alpha_{11}^2 Q_1^\varepsilon(z) + 2\alpha_{11}\alpha_{12} Q_2^\varepsilon(z) + \alpha_{12}^2 Q_3^\varepsilon(z),\\
f_2^\varepsilon(z) &= \big(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}\big) Q_2^\varepsilon(z) + \alpha_{11}\alpha_{21} Q_1^\varepsilon(z) + \alpha_{12}\alpha_{22} Q_2^\varepsilon(z),\\
f_3^\varepsilon(z) &= \alpha_{21}^2 Q_1^\varepsilon(z) + 2\alpha_{21}\alpha_{22} Q_2^\varepsilon(z) + \alpha_{22}^2 Q_3^\varepsilon(z),
\end{aligned}$$

then

$$z_1^2 = \lim_{\varepsilon \longrightarrow 0} f_1^\varepsilon(z) \in \liminf_{\varepsilon \longrightarrow 0} \mathcal{I}_\varepsilon;$$
$$z_1 z_2 = \lim_{\varepsilon \longrightarrow 0} f_2^\varepsilon(z) \in \liminf_{\varepsilon \longrightarrow 0} \mathcal{I}_\varepsilon;$$
$$z_2^2 = \lim_{\varepsilon \longrightarrow 0} f_3^\varepsilon(z) \in \liminf_{\varepsilon \longrightarrow 0} \mathcal{I}_\varepsilon,$$

which proves that $\mathfrak{M}_0^2 \subset \liminf_\varepsilon \mathcal{I}_\varepsilon$, and thus that condition (1.2) is sufficient for the claimed convergence.

2.3. **Proof of Theorem 1.5.** Statement (2) in Theorem 1.5 is a special case of the proof above. We now turn to statement (1).

We slightly modify the varying basis $\mathfrak{B}_\varepsilon$ from the previous proof. The hypothesis of Theorem 1.5 implies that $\lim_\varepsilon [e_1^\varepsilon]$ exists, so multiplying $e_1^\varepsilon, e_2^\varepsilon$ by appropriate complex numbers of modulus one, we get a basis $\tilde{\mathfrak{B}}_\varepsilon = (\tilde{e}_1^\varepsilon, \tilde{e}_2^\varepsilon)$ such that $\lim_\varepsilon \tilde{e}_1^\varepsilon =: e_1$ and $\lim_\varepsilon \tilde{e}_2^\varepsilon =: e_2$ exist. We denote by $(z_1^\varepsilon, z_2^\varepsilon)$, respectively (z_1, z_2), the coordinates in $(\tilde{e}_1^\varepsilon, \tilde{e}_2^\varepsilon)$, resp. (e_1, e_2).

Finally, given any function f expressed in the (z_1, z_2)-coordinates, we denote by $\tilde{f}$ the function computed in the $(z_1^\varepsilon, z_2^\varepsilon)$-coordinates (i.e. if z and $\tilde{z}$ are the coordinates of the same point, $f(z) = \tilde{f}(\tilde{z})$). We write

$$\tilde{f}^\varepsilon(z_1^\varepsilon, z_2^\varepsilon) = \sum_{j,k} \tilde{a}_{ij}^\varepsilon (z_1^\varepsilon)^j (z_2^\varepsilon)^k$$

(both the function and the coordinates depend on ε). Since this is a linear change of variables, the various Taylor coefficients are obtained from the chain rule by linear formulae, and since the change of variables matrix tends to the identity as $\varepsilon \to 0$, $\lim_\varepsilon a_{ij}^\varepsilon = \lim_\varepsilon \tilde{a}_{ij}^\varepsilon$ if the latter exists.

Again, let $f \in \liminf_\varepsilon \mathcal{I}_\varepsilon$, i.e. $f = \lim_\varepsilon f^\varepsilon$ with uniform convergence on a fixed neighborhood of the origin, $f^\varepsilon \in \mathcal{I}_\varepsilon$. Applying Lemma 2.2 for $m = 2$, taking $a_{0,0}^\varepsilon = 0$ into account,

$$\tilde{f}^\varepsilon(z_1, z_2) = \tilde{a}_{1,0}^\varepsilon z_1 + \tilde{a}_{0,1}^\varepsilon z_2 + \tilde{a}_{2,0}^\varepsilon z_1^2 + \tilde{a}_{0,2}^\varepsilon z_2^2 + \tilde{a}_{1,1}^\varepsilon z_1 z_2 + R_3(z_1, z_2)$$

with $|R_3(z_1, z_2)| \le C\|z\|^3$.

Since $\tilde{f}^\varepsilon(a_2^\varepsilon) = \tilde{f}^\varepsilon(\varepsilon, 0) = 0$ we have $\tilde{a}_{1,0}^\varepsilon \varepsilon + \tilde{a}_{2,0}^\varepsilon \varepsilon^2 + R_3(\varepsilon, 0) = 0$. Thus

$$\tilde{a}_{1,0}^\varepsilon = -\tilde{a}_{2,0}^\varepsilon \varepsilon - \frac{R_3(\varepsilon, 0)}{\varepsilon} \tag{2.2}$$

for any ε.

Thus $\dfrac{\partial \tilde{f}}{\partial z_1}(0,0) = \tilde{a}_{1,0} = \lim_{\varepsilon \to 0} \tilde{a}_{1,0}^\varepsilon = 0$.

Furthermore, setting $\rho=\rho(\varepsilon), \delta=\delta(\varepsilon)$, from $\tilde f^\varepsilon(a_3^\varepsilon)=\tilde f^\varepsilon(\rho,\delta\rho)=0$, and (2.2) we deduce

$$\begin{aligned}\Big[-\tilde a^\varepsilon_{2,0}\varepsilon-\frac{R_3(\varepsilon,0)}{\varepsilon}\Big]\rho+\tilde a^\varepsilon_{0,1}\delta\rho+\tilde a^\varepsilon_{2,0}\rho^2+\tilde a^\varepsilon_{0,2}\delta^2\rho^2\\ +\tilde a^\varepsilon_{1,1}\delta\rho^2+R_3(\rho,\delta\rho)=0,\end{aligned}\tag{2.3}$$

and dividing by $\rho(\rho-\varepsilon)$,

$$\tilde a^\varepsilon_{2,0}+\tilde a^\varepsilon_{0,1}\frac{\delta}{\rho-\varepsilon}+\tilde a^\varepsilon_{0,2}\delta^2\frac{\rho}{\rho-\varepsilon}+\tilde a^\varepsilon_{1,1}\frac{\delta}{\rho-\varepsilon}\rho+\frac{R_3(\rho,\delta\rho)}{\rho(\rho-\varepsilon)}-\frac{R_3(\varepsilon,0)}{\varepsilon(\rho-\varepsilon)}=0.$$

Note that since $|\rho(\varepsilon)|\leqslant\frac{1}{2}|\varepsilon|$, $\frac{2}{3}\leqslant\left|\frac{\varepsilon}{\rho-\varepsilon}\right|\leqslant 2$, and $\left|\frac{\rho}{\rho-\varepsilon}\right|=\left|\frac{\varepsilon}{\rho-\varepsilon}\right|\left|\frac{\rho}{\varepsilon}\right|\leqslant 2\left|\frac{\rho}{\varepsilon}\right|\leqslant 1$.

Since $R_3(\rho,\delta\rho)=O(\rho^3)$, $\lim\limits_{\varepsilon\longrightarrow 0}\frac{R_3(\rho,\delta\rho)}{\rho(\rho-\varepsilon)}=0$ and passing to the limit as explained above, $a_{2,0}+ma_{0,1}=0$. Thus

$$\limsup_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon\subset\mathcal{I}:=\Big\{f\in\mathcal{O}(\mathbb{D}^2): f(0,0)=\frac{\partial f}{\partial z_1}(0,0)=0 \text{ and } \frac{1}{2}\frac{\partial^2 f}{\partial z_1^2}(0,0)+m\frac{\partial f}{\partial z_2}(0,0)=0\Big\}=\left\langle z_2-mz_1^2, z_2^2, z_1z_2, z_1^3\right\rangle.$$

Since $z_1z_2=z_1(z_2-mz_1^2)+mz_1^3$ et $z_2^2=(z_2+mz_1^2)(z_2-mz_1^2)+(m^2z_1)z_1^3$, we have $z_1z_2, z_2^2\in\langle z_2-mz_1^2, z_1^3\rangle$. Thus $\mathcal{I}=\langle z_2-mz_1^2, z_1^3\rangle$.

Conversely,

$$z_2-mz_1^2=\lim_{\varepsilon\longrightarrow 0}\left(z_2^\varepsilon-\frac{\delta}{\rho-\varepsilon}z_1^\varepsilon(z_1^\varepsilon-\varepsilon)\right)\in\liminf_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon,\text{ and}$$

$$z_1^3=\lim_{\varepsilon\longrightarrow 0}z_1^\varepsilon(z_1^\varepsilon-\varepsilon)\big(z_1^\varepsilon-\rho(\varepsilon)\big)\in\liminf_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon.$$

Thus $\mathcal{I}\subset\liminf\limits_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon$. We have proved $\lim\limits_{\varepsilon\longrightarrow 0}\mathcal{I}_\varepsilon=\mathcal{I}=\langle z_2-mz_1^2, z_1^3\rangle$.

Since $\mathcal{I}$ admits a representation by two generators, the second statement in (1) follows from Theorem 1.3.

2.4. **Proof of the necessity in Theorem 1.4.** If $\frac{\|a_2^\varepsilon\|}{|\theta|}$ does not tend to 0, we can find a sequence $\varepsilon_j\to 0$ such that $\frac{\|a_2^{\varepsilon_j}\|}{|\theta|}\geqslant c>0$ and therefore along this sequence θ tends to 0, so the distances in $\mathbb{CP}^1$ between all $[a_i^\varepsilon-a_j^\varepsilon]$ tend to 0. Also, $\theta\sim\tan\theta\sim\sin\theta$. Passing to a further subsequence, we may assume that $a_2^\varepsilon/\|a_2^\varepsilon\|$ converges. Using the coordinates and notations of Theorem 1.5, we have that $\frac{\delta(\varepsilon_j)}{\rho(\varepsilon_j)-\varepsilon_j}$ is a

bounded sequence in $\mathbb{C}$. Then, passing to another subsequence, we may assume that $\lim_j \dfrac{\delta(\varepsilon_j)}{\rho(\varepsilon_j)-\varepsilon_j} = m \in \mathbb{C}$ and so we are in the situation of Theorem 1.5, statement (1). So the limit of the ideals $\mathcal{I}_\varepsilon$ along this subsequence contains the function z_2 (given by this appropriate coordinate system), which implies that $\limsup_\varepsilon \mathcal{I}_\varepsilon \not\subset \mathfrak{M}_0^2$.

We remark that in this case, if $[a_2^\varepsilon]$ does not converge, then we can find two different limit values for it, and two different functions of degree 1 in $\limsup_\varepsilon \mathcal{I}_\varepsilon$, so that $\limsup_\varepsilon \mathcal{I}_\varepsilon = \mathfrak{M}_0$ and (for reasons of length) the family $(\mathcal{I}_\varepsilon)$ cannot converge to any limit ideal. This is in contrast to the other case, where no convergence of the varying basis $\mathfrak{B}_\varepsilon$ was required.

3. Proof of Proposition 1.6

The definition of the Green function implies that for any $f \in \mathcal{O}(\mathbb{D}^2)$, with $\|f\|_\infty \leq 1$, such that $f(a_j^\varepsilon) = 0$, $1 \leq j \leq 3$, then $G_\varepsilon(z) \geq \log|f(z)|$.

First consider the polynomial $Q(z_1,z_2) = -\varepsilon z_1 + z_1^2 + \dfrac{\varepsilon-\rho}{\delta} z_2$. Define

$$Q_1(z_1,z_2) := \left(\sup_{\substack{|w_1|<1\\ |w_2|<1}} \left| Q(w_1,w_2) \right| \right)^{-1} Q(z_1,z_2).$$

It is easy to see that it satisfies the conditions above, so letting ε tend to 0,

$$\liminf_{\varepsilon \longrightarrow 0} G_\varepsilon(z_1,z_2) \geqslant \log|z_1|^2 = 2\log|z_1|, \tag{3.1}$$

for any $z = (z_1,z_2) \in \mathbb{D}^2 \setminus \{(0,0)\}$.

To get the other part of the estimate, consider the three lines passing through two points a_i^ε and a_j^ε, $(i \neq j, i, j \in \{1,2,3\})$, with the following equations :

$$l_1^\varepsilon(z) = z_2; \quad l_2^\varepsilon(z) = z_2 - \delta(\varepsilon) z_1; \quad l_3^\varepsilon(z) = z_2 - \delta(\varepsilon)\frac{\rho(\varepsilon)}{\rho(\varepsilon)-\varepsilon}(z_1-\varepsilon).$$

Since any pole belong to two of the lines, $P(z_1,z_2) := l_1^\varepsilon(z)\cdot l_2^\varepsilon(z)\cdot l_3^\varepsilon(z) \in \mathcal{I}_\varepsilon^2$. So

$$G_\varepsilon(z_1,z_2) \geqslant \frac{1}{2}\log\frac{|P(z_1,z_2)|}{||P||_\infty}.$$

Furthermore

$$||P||_\infty \leqslant 1(1+|\delta|)(1+|\delta|.(1+|\varepsilon|)),$$

so $\lim_{\varepsilon\to 0} ||P||_\infty = 1$. Letting ε tend to 0,

$$\liminf_{\varepsilon\longrightarrow 0} G_\varepsilon(z_1,z_2) \geqslant \frac{3}{2}\log|z_2|,$$

for any $z=(z_1,z_2)\in\mathbb{D}^2\setminus\{(0,0)\}$.

To obtain Part (2) of the Proposition, we observe as in [4] that

$$G_\varepsilon(z) \le G_{\{a_1^\varepsilon,a_2^\varepsilon\}}(z) = \max\left(\log\left|z_1\frac{\varepsilon - z_1}{1-\bar\varepsilon z_1}\right|, \log|z_2|\right),$$

and this last function tends to $2\log|z_1|$ when $|z_2|\le|z_1|^2$. □

4. Proof of Proposition 1.7

We will estimate the function by restricting it to well-chosen families of analytic disks. By Proposition 1.6 (2), there is no loss in assuming $z_2\neq 0$.

Let $Z_1 := \dfrac{z_1-\varepsilon}{\|z\|_\infty} r_\varepsilon$, $Z_2 := \dfrac{z_2}{\|z\|_\infty} r_\varepsilon$, where $r_\varepsilon := 1-\dfrac{|\varepsilon||z_1|}{|z_1|-|\varepsilon|}\le 1$. If $z_1 = 1$, $r_\varepsilon = 1$; if $z_1\neq 0$, r_ε is defined for $|\varepsilon|<|z_1|$ and in this case $0<r_\varepsilon\to 1$ as $\varepsilon\to 0$. Note that $|Z_1|,|Z_2|\le 1$. Set

$$\Psi(\zeta) := (\varepsilon + Z_1\zeta, Z_2\zeta) = \left(\varepsilon + \frac{z_1-\varepsilon}{\|z\|_\infty} r_\varepsilon\zeta, \frac{z_2}{\|z\|_\infty} r_\varepsilon\zeta\right).$$

For $\zeta\in\mathbb{D}$, we have $\left|\dfrac{z_2}{\|z\|_\infty} r_\varepsilon\zeta\right| < r_\varepsilon \le 1$ and $\left|\varepsilon + \dfrac{z_1-\varepsilon}{\|z\|_\infty} r_\varepsilon\zeta\right| < 1$. Since, when $z_1\neq 0$,

$$\left|r_\varepsilon\zeta + \frac{\varepsilon\|z\|_\infty}{z_1-\varepsilon}\right| < |\zeta|\left(1-\frac{|\varepsilon||z_1|}{|z_1|-|\varepsilon|}\right) + \frac{|\varepsilon||z_1|}{|z_1|-|\varepsilon|} < 1.$$

Thus $\Psi(\zeta)\in\mathbb{D}^2$. Furthermore, for $|\varepsilon|<\varepsilon_0$, we have $\left|\dfrac{\|z\|_\infty}{r_\varepsilon}\right| < 1$ et $\Psi\big(\dfrac{\|z\|_\infty}{r_\varepsilon}\big) = (z_1,z_2)$, $\Psi(0) = a_1^\varepsilon = (\varepsilon, 0)$.

Let $u\in PSH_{\text{-}}(\mathbb{D}^2)$ a function in the defining family of G_ε. Set $u_2 := u\circ\Psi$. For $\zeta\in\mathbb{D}$,

$$u_2(\zeta) = u\big(\varepsilon + Z_1\zeta, Z_2\zeta\big) \leqslant \log\max\big(|Z_1\zeta|,|Z_2\zeta|\big) + O(1) \leqslant \log|\zeta| + O(1).$$

So write $u_3(\zeta) := u_2(\zeta) - \log|\zeta|$, for $\zeta\in\mathbb{D}\setminus\{0\}$. Since $u_2(\zeta)\in SH_{\text{-}}\big(\mathbb{D}\setminus\{0\}\big)$ and $\log|\zeta|\in H\big(\mathbb{D}\setminus\{0\}\big)$, $u_3\in SH_{\text{-}}\big(\mathbb{D}\setminus\{0\}\big)$. Near 0, $u_3(\zeta) = u_2(\zeta) - \log|\zeta| \leqslant \log|\zeta| + O(1) - \log|\zeta| = O(1)$, so u_3 is bounded in a neighborhood of 0. By the removable singularity theorem for subharmonic functions [5, Theorem 3.6.1], we can extend u_3 to a function in $SH_{\text{-}}\big(\mathbb{D}\big)$.

Lemma 4.1. *For any $u \in PSH_{-}(\mathbb{D}^2)$ a function in the defining family of G_ε, there exist $z_0 \in \mathbb{D}$ and constants $C_1, C_4 > 0$ such that for any $\zeta \in D_0 := D(z_0, C_1|\varepsilon|^2)$,*

$$u_3(\zeta) \leq \log\left|\frac{\varepsilon}{\delta}\right| + C_4.$$

Let us postpone the proof of this lemma (which will use the other two poles and another family of analytic discs). Notice that it doesn't use the hypothesis $\lim_{\varepsilon \longrightarrow 0} \dfrac{\log|\delta|}{\log|\varepsilon|} = 0$.

For any $\xi \in \mathbb{D}$, set $u_4(\xi) := u_3 \circ \Phi_{\xi(0)}(\xi)$, where $\Phi_{\xi(0)}(\xi) := \dfrac{\xi(0) - \xi}{1 - \overline{\xi(0)}\xi}$ is the standard Möbius involution of the disk. Then $u_4 \in SH_{-}(\mathbb{D})$. Set $D_1 := D\big(0, C_5|\varepsilon|^2\big) \subset \Phi^{-1}_{\xi(0)}(D_0)$. For any $\eta \in \overline{D_1}$, let $\xi = \Phi_{\xi(0)}(\eta) \in \overline{D_0}$. Then (4.8) implies $u_4(\eta) = u_3(\xi) \leqslant \log\left|\dfrac{\varepsilon}{\delta}\right| + C_4$.

By the three-circle property for subharmonic functions, for $\xi \in \overline{\mathbb{D}} \backslash D_1$, we have

$$\begin{aligned}
u_4(\xi) &\leqslant \left(\log\left|\frac{\varepsilon}{\delta}\right| + C_3\right) \cdot \frac{\log|\xi|}{\log\big(C_4|\varepsilon|^2\big)} \\
&= \log|\xi| \cdot \left(\frac{1 - \dfrac{\log|\delta|}{\log|\varepsilon|} + \dfrac{C_3}{\log|\varepsilon|}}{2 + \dfrac{\log C_4}{\log|\varepsilon|}}\right).
\end{aligned} \tag{4.1}$$

For any $\xi \in \overline{\mathbb{D}} \backslash D_0$, $\Phi^{-1}_{\xi(0)}(\xi) = \Phi_{\xi(0)}(\xi) \subset \overline{\mathbb{D}} \backslash D_1$. From (4.1), we get

$$\begin{aligned}
u_2(\xi) &= u_4\big(\Phi_{\xi(0)}(\xi)\big) + \log|\xi| \\
&\leqslant \log|\xi| + \log|\Phi_{\xi(0)}(\xi)| \cdot \left(\frac{1 - \dfrac{\log|\delta|}{\log|\varepsilon|} + \dfrac{C_3}{\log|\varepsilon|}}{2 + \dfrac{\log C_4}{\log|\varepsilon|}}\right).
\end{aligned}$$

On the other hand, for ε small enough, $\left|\dfrac{\|z\|_\infty}{r_\varepsilon} - \xi(0)\right| \geqslant r_0 = C|\varepsilon|^2$.
Pick $\xi = \dfrac{\|z\|_\infty}{r_\varepsilon} \in \mathbb{D}\backslash D_0$, then

$$
\begin{aligned}
u(z_1, z_2) &= u\left(\Psi\left(\frac{\|z\|_\infty}{r_\varepsilon}\right)\right) = u_2\left(\frac{\|z\|_\infty}{r_\varepsilon}\right) \\
&\leqslant \log\left(\frac{\|z\|_\infty}{r_\varepsilon}\right) + \log\left|\mathit{\Phi}_{\xi(0)}\left(\frac{\|z\|_\infty}{r_\varepsilon}\right)\right| . \left(\frac{1 - \dfrac{\log|\delta|}{\log|\varepsilon|} + \dfrac{C_3}{\log|\varepsilon|}}{2 + \dfrac{\log C_4}{\log|\varepsilon|}}\right).
\end{aligned}
$$

Letting ε tend to 0,

$$
\lim_{\varepsilon \longrightarrow 0} \log\left|\mathit{\Phi}_{\xi(0)}\left(\frac{\|z\|_\infty}{r_\varepsilon}\right)\right| = \log\|z\|_\infty,
$$

for any $z = (z_1, z_2) \in \mathbb{D}^2\backslash\{z_2 = 0\}$.

We now use the hypothesis $\lim\limits_{\varepsilon \longrightarrow 0} \dfrac{\log|\delta|}{\log|\varepsilon|} = 0$, and get $\limsup\limits_{\varepsilon \longrightarrow 0} u(z) \leqslant \log\|z\|_\infty + \dfrac{1}{2}\log\|z\|_\infty = \dfrac{3}{2}\log\|z\|_\infty$, for any $z = (z_1, z_2) \in \mathbb{D}^2\backslash\{z_2 = 0\}$.
□

Proof of Lemma 4.1.

First we define a family of analytic disks going through $a_0^\varepsilon = (0, 0)$ and $a_2^\varepsilon = (\rho, \delta\rho)$. For $\zeta \in \mathbb{D}$, let

$$
\varphi_\lambda(\zeta) := \big(\zeta, \delta\zeta + \lambda\zeta(\zeta - \rho)\big),
$$

where

$$
|\lambda| \leqslant \frac{1 - |\delta|}{1 + |\rho|} < 1. \tag{4.2}
$$

We have $\varphi_\lambda(\zeta) \in \mathbb{D}^2$ for any $\zeta \in \mathbb{D}$ because of

$$
\big|\delta\zeta + \lambda\zeta(\zeta - \rho)\big| \leqslant |\delta| + |\lambda|\big(1 + |\rho|\big) \leqslant |\delta| + \frac{1 - |\delta|}{1 + |\rho|}.\big(1 + |\rho|\big) = 1.
$$

Let $u_1(\zeta) := u \circ \varphi_\lambda(\zeta) = u\big(\zeta, \delta\zeta + \lambda\zeta(\zeta - \rho)\big)$. From the definition of the Green function, we have

$$
u_1(\zeta) \leqslant \log\max\big(|\zeta|, |\zeta|\big|\delta + \lambda(\zeta - \rho)\big|\big) + O(1) = \log|\zeta| + O(1),
$$

because $\big|\delta + \lambda(\zeta - \rho)\big| \leqslant |\delta| + |\lambda|\big(1 + |\rho|\big) \leqslant 1$. Furthermore,

$$
\begin{aligned}
u_1(\zeta) &= u\big(\rho + \zeta - \rho, \delta\rho + \delta(\zeta - \rho) + \lambda\zeta(\zeta - \rho)\big) \\
&\leqslant \log\max\big(\big|\zeta - \rho\big|, \big|\zeta - \rho\big| \cdot \big|\delta + \lambda\zeta\big|\big) + O(1) \leqslant \log\big|\zeta - \rho\big| + O(1).
\end{aligned}
$$

Therefore for $\zeta \in \mathbb{D}$

$$u_1(\zeta) \leqslant G^{\mathbb{D}}_{\{0,\rho\}}(\zeta) = \log \left|\zeta \frac{\rho - \zeta}{1 - \overline{\rho}\zeta}\right|. \tag{4.3}$$

This provides a certain upper bound for the values of u on the union of the ranges of the disks φ_λ. We want to see how it will affect u_2, the restriction of u on the straight disk $\Psi(\mathbb{D})$ going through $a_1^\varepsilon = (\varepsilon, 0)$ and z.

We look for $\zeta = \zeta(\lambda) \in \mathbb{D}$ such that $\varphi_\lambda(\zeta) = \Psi(\xi) \in \Psi(\mathbb{D})$. Then $(\varepsilon + Z_1\xi, Z_2\xi) = \big(\zeta, \delta\zeta + \lambda\zeta(\zeta - \rho)\big)$, thus $\zeta = \varepsilon + Z_1\xi$ and substituting into the equation for the second coordinates,

$$Z_2\xi = \delta(\varepsilon + Z_1\xi) + \lambda(\varepsilon + Z_1\xi)(\varepsilon - \rho + Z_1\xi). \tag{4.4}$$

For $z_2 \neq 0$, $Z_2 - \delta Z_1 \neq 0$ for $|\varepsilon| < \varepsilon_0$. If $\lambda = 0$ the solution of (4.4) is

$$\xi(0) := \frac{\delta\varepsilon}{Z_2 - \delta Z_1}.$$

If $\lambda \neq 0$, let us write the solution of (4.4)) in the following form: $\xi = \xi(\lambda) = \xi(0) + \beta(\lambda) = \dfrac{\delta\varepsilon}{Z_2 - \delta Z_1} + \beta(\lambda)$; $\beta := \beta(\lambda)$. Then (4.4) transforms into

$$(Z_2 - \delta Z_1)\beta = \lambda\left(\varepsilon + \frac{Z_1\delta\varepsilon}{Z_2 - \delta Z_1} + Z_1\beta\right)\left(\varepsilon - \rho + \frac{Z_1\delta\varepsilon}{Z_2 - \delta Z_1} + Z_1\beta\right). \tag{4.5}$$

This equation is of the form

$$a\beta^2 + (\theta_1 - b_0)\beta + c = 0 \tag{4.6}$$

where $a := \lambda Z_1^2$, $b_0 = Z_2 - \delta Z_1$, $\theta_1 = O(\varepsilon)$, $c = O(\varepsilon^2)$. The solutions of (4.6) are

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{\big(b_0 - \theta_1\big) \pm \sqrt{b_0^2 - 2b_0\theta_1 + \theta_1^2 - 4ac}}{2a} \\ &= \frac{b_0}{2a}\left[1 - \frac{\theta_1}{b_0} \pm \sqrt{1 - \frac{2\theta_1}{b_0} + \frac{\theta_1^2}{b_0^2} - \frac{4ac}{b_0^2}}\right].\end{aligned}$$

One of them satisfies

$$\beta(\lambda) = \frac{b_0}{2a}\left[O\left(\frac{\theta_1^2}{2b_0^2} - \frac{2ac}{b_0^2}\right)\right] = O(\varepsilon^2).$$

On the other hand, (4.5) implies

$$\lambda = \frac{\beta\big(Z_2 - \delta Z_1\big)}{\left(\varepsilon + \dfrac{Z_1\delta\varepsilon}{Z_2 - \delta Z_1} + Z_1\beta\right)\left(\varepsilon - \rho + \dfrac{Z_1\delta\varepsilon}{Z_2 - \delta Z_1} + Z_1\beta\right)}.$$

Since $|\varepsilon-\rho| \geqslant \dfrac{1}{2}|\varepsilon|$ and $\big|Z_2-\delta Z_1\big| \leqslant 1+|\delta|$, it follows that

$$|\lambda| = \frac{\big|(Z_2-\delta Z_1)\big||\beta|}{\Big|\big(\varepsilon+O(|\delta\varepsilon|)\big)\big(\varepsilon-\rho+O(|\delta\varepsilon|)\big)\Big|} \leqslant \frac{|\beta|\big(1+|\delta|\big)}{1/2|\varepsilon|^2}.$$

So $\dfrac{|\beta|\big(1+|\delta|\big)}{1/2|\varepsilon|^2} \leqslant \dfrac{1-|\delta|}{1+|\rho|}$, i.e.

$$|\beta(\lambda)| \leqslant \frac{1-|\delta|}{1+|\rho|}\cdot\frac{1}{1+|\delta|}\cdot\frac{1}{2}|\varepsilon|^2 < \frac{1}{2}|\varepsilon|^2,$$

and $|\lambda| \leqslant \dfrac{1-|\delta|}{1+|\rho|}$.

Set $z_0 := \xi(0) = \dfrac{\delta\varepsilon}{Z_2-\delta Z_1}$. For $|\varepsilon|$, therefore $|\delta|$ small enough, $|z_0| = \left|\dfrac{\delta\varepsilon}{Z_2-\delta Z_1}\right| \leqslant C_0(z)|\varepsilon| < 1$. Now consider a disc $D_0 := D(z_0; r_0)$, where $r_0 := \frac{1}{2}|\varepsilon|^2$.

Since $\xi \in \overline{D_0}$, $|\zeta(\xi)| = |\varepsilon + Z_1\xi| \leqslant |\varepsilon| + |\xi| \leqslant (|\varepsilon| + C_0|\varepsilon| + \frac{1}{2}|\varepsilon|^2) \leqslant C_1|\varepsilon|$, where $C_0 := C_0(z)$ and $0 < C_1$ do not depend on ε. So

$$\begin{aligned}\left|(\varepsilon+Z_1\xi)\frac{\rho-\varepsilon-Z_1\xi}{1-\overline{\rho}(\varepsilon+Z_1\xi)}\right| &\leqslant C_1|\varepsilon|.\frac{1/2|\varepsilon|+C_1|\varepsilon|}{\big|1-1/2C_1|\varepsilon|^2\big|} \\ &\leqslant C_1\frac{1/2+C_1}{|1-1/2C_1|}|\varepsilon|^2 \leqslant C_2|\varepsilon|^2,\end{aligned} \tag{4.7}$$

where $0 < C_2$ does not depend on ε. So for $|\varepsilon|$ small enough and at least $\leqslant \dfrac{1}{C_1}$, there is $\zeta(\xi) \in \mathbb{D}$ such that $\Psi(\xi) = \varphi_\lambda\big(\zeta(\xi)\big)$. Thus

$$u_2(\xi) = u\big(\Psi(\xi)\big) = u\big(\varphi_\lambda\big(\zeta(\xi)\big)\big) = u_1(\zeta(\xi)) = u_1(\varepsilon+Z_1\xi).$$

From (4.3) and (4.7), we deduce

$$\begin{aligned}u_1(\varepsilon+Z_1\xi) \leqslant G^{\mathbb{D}}_{\{0,\rho\}}(\varepsilon+Z_1\xi) &= \log\left|(\varepsilon+Z_1\xi)\frac{\rho-\varepsilon-Z_1\xi}{1-\overline{\rho}(\varepsilon+Z_1\xi)}\right| \\ &\leqslant \log|\varepsilon|^2 + \log C_2.\end{aligned}$$

For any $\xi \in \overline{D_0}$,

$$||z_0|| = \frac{\big|\delta\varepsilon\big|}{\big|Z_2-\delta Z_1\big|} > \frac{|\delta||\varepsilon|}{\big|Z_2-\delta Z_1\big|} > \frac{|\delta||\varepsilon|}{1+|\delta|} > \frac{1}{2}|\varepsilon|^2 = r_0,$$

for $|\varepsilon|$ small enough depending on z, so $\xi \neq 0$ and

$$u_3(\xi) = u_2(\xi) - \log|\xi| \leqslant \log|\varepsilon|^2 - \log|\xi| + \log C_2.$$

In addition, $\big|\xi - \xi(0)\big| = |\beta(\lambda)| < 1/2|\varepsilon|^2$ and $|\varepsilon\delta| \gg |\varepsilon|^2$ imply that

$$|\xi| \geqslant \frac{1}{\big|Z_2 - \delta Z_1\big|}|\varepsilon\delta| - \frac{1}{2}|\varepsilon|^2 > \left(\frac{1}{\big|Z_2 - \delta Z_1\big|} - \frac{1}{2}\right)|\delta\varepsilon| > 0$$

(because $\big|Z_2 - \delta Z_1\big| < 1 + |\delta| < 2$). Then

$$u_3(\xi) \leqslant \log|\varepsilon|^2 - \log|\varepsilon\delta| - \log\left(\frac{1}{\big|Z_2 - \delta Z_1\big|} - \frac{1}{2}\right) + \log C_2,$$

for any $\xi \in \overline{D_0}$. Finally

(4.8)

$$u_3(\xi) \leqslant \log\left|\frac{\varepsilon}{\delta}\right| + C_3, \text{ with } C_3 := C_3(z) = -\log\left(\frac{1}{\big|Z_2 - \delta Z_1\big|} - \frac{1}{2}\right) + \log C_2.$$

□

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